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BIMODAL OPTIMAL DESIGN OF CLAMPED–CLAMPED COLUMNS UNDER CREEP CONDITIONS†

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Abstract—The problem of determining the optimal cross-sectional area function of clamped–clamped columns under creep conditions is investigated by using the Pontryagin maximum principle. Total volume V is minimized under given axial force P, critical time in the Rabotnov–Shesterikov sense t_* and additional geometrical constraints.

1. INTRODUCTION

There exist various theories of creep buckling of columns. Some of them analyze deflections of imperfect columns and define the critical time as corresponding to infinite deflections or infinite deflection rates of such structures (Kempner, Hoff and Fraeijs de Veubeke). In the present paper, however, we discuss optimization of perfectly straight and axially compressed columns. The critical time is then defined as that corresponding to loss of stability of the creep process in uniaxial compression. First proposals of such an approach are due to Shanley[1] and Gerard[2], but it was Rabotnov and Shesterikov[3] who derived a more consistent theory of that kind. A comparison of the above theories was performed by Jahsman and Field[4]. Though certain objections were raised by Jahsman and Field, and by Hoff[5, 6], it is the Rabotnov–Shesterikov creep buckling theory which will serve as the basis for the represent considerations. It seems that a change of the theory would influence the optimal shapes only very slightly.

Optimization of simply supported columns under creep conditions according to the Rabotnov–Shesterikov theory was analyzed by Życzkowski and Wojdanowska-Zając[7]. However, in some cases of support, e.g. for clamped–clamped columns, a unimodal (single) approach to optimal design leads to erroneous results and should be replaced by a bimodal one. Bimodal otpimization, mentioned by Kiusalaas for columns on elastic foundation[8] was discussed in detail for clamped–clamped columns by Olhoff and Rasmussen[9]. The present paper may be considered as a generalization of [7, 9–11]. The method of solution will be based on Pontryagin's maximum principle.

2. STATE EQUATIONS

Let us consider a nonprismatic column of length l, compressed by the force P. Following Rabotnov–Shesterikov we assume that the material is governed by the equation:

$$\tilde{\Phi} = \dot{p}p^{\alpha} - A\sigma^{n} = 0, p = \epsilon - \frac{\sigma}{E} \tag{1}$$

where A, n and α denote material constants depending on the temperature, while σ and p are stress and inelastic strain. In the state of pure compression before buckling one obtains the following expression for p:

$$p = (\alpha + 1)^{1/(\alpha+1)} A^{1/(\alpha+1)} \sigma^{n/(\alpha+1)} t^{1/(\alpha+1)}. \tag{2}$$

Expressing p in terms of $p = p(\sigma, \epsilon)$ and supposing that at the instant of time $t = t_*$ the equilibrium in the neighbouring position may exist (critical time), the following equation

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is obtained:

$$(E\lambda_* - \gamma_*)M - EI\gamma_* w'' = 0 \tag{3}$$

where M, I stand for the bending moment and the moment of inertia of the cross-section, respectively. Evaluating the coefficients λ_*, γ_*, we obtain[7]

$$\frac{EIw''}{1 + \frac{n}{\alpha} E[A(\alpha+1)t_*]^{1/(\alpha+1)}[F(x)]^{(-n+\alpha+1)/(\alpha+1)} P^{(n-\alpha-1)/(\alpha+1)}} = -M \tag{4}$$

where $F(x)$ is the cross-sectional area. For a clamped–clamped column the bending moment M depends on redundant reactive forces. To eliminate them we differentiate (4) twice with respect to x, obtaining

$$\left\{\frac{EIw''}{1 + \frac{n}{\alpha} E[A(\alpha+1)t_*]^{1/(\alpha+1)}\left[\frac{P}{F(x)}\right]^{(n-\alpha-1)/(\alpha+1)}}\right\}'' + Pw'' = 0. \tag{5}$$

Let us introduce the following dimensionless quantities:

$$I = I_0 \Phi^{\nu},\ \Phi(x) = F(x)/F_0,\ w = y/l,\ \beta = Pl^2/EI_0 \tag{6}$$

where $\nu = 2$ for similar cross-sections, $\nu = 1$ or $\nu = 3$ for plane-tapered columns. Then the eqn (5) can be rewritten as:

$$\left\{\frac{\Phi^{\nu} y''}{1 + \frac{n}{\alpha} E[A(\alpha+1)t_*]^{1/(\alpha+1)}\left(\frac{\beta}{\Phi}\right)^{(n-\alpha-1)/(\alpha+1)}\left(\frac{F_0 l^2}{EI_0}\right)^{(-n+\alpha+1)/(\alpha+1)}}\right\}'' + \beta y'' = 0 \tag{7}$$

or in a more compact form:

$$\left\{\frac{\Phi^{\nu} y''}{1 + T\left(\frac{\beta}{\Phi}\right)^{(n-\alpha-1)/(\alpha+1)}}\right\}'' + \beta y'' = 0 \tag{8}$$

where the following parameter T has been defined:

$$T = \frac{n}{\alpha}[A(\alpha+1)t_*]^{1/(\alpha+1)} E^{n/(\alpha+1)}\left(\frac{F_0 l^2}{I_0}\right)^{(-n+\alpha+1)/(\alpha+1)} \tag{9}$$

F_0 and I_0 are the cross-sectional area and the second moment of area, respectively, at the point x_0 for which the volume of the column is $V = F_0 l$.

Equation (8) is broken into a set of four first order differential equations as it is customary in the formalism of problems of optimal control. In what follows they are called the state equations:

$$\begin{aligned}
y_i' &= \varphi_i \\
\varphi_i' &= -\frac{1 + T\left(\frac{\beta_i}{\Phi}\right)^{(n-\alpha-1)/(\alpha+1)}}{\Phi^{\nu}} M_i \\
M_i' &= Q_i + \beta_i \varphi_i \\
Q_i' &= 0.
\end{aligned} \tag{10}$$

The subscript $i = 1$ refers to the antisymmetric form of buckling, while $i = 2$ to the symmetric one. The boundary conditions have the form:

(a) the antisymmetric form ($i = 1$)

$$\begin{aligned} y_1(0) &= 0, \qquad & \varphi_1(0) &= 0 \\ y_1(1/2) &= 0, \qquad & M_1(1/2) &= 0. \end{aligned} \tag{11}$$

(b) the symmetric form ($i = 2$)

$$\begin{aligned} y_2(0) &= 0, \qquad & \varphi_2(0) &= 0 \\ \varphi_2(1/2) &= 0, \qquad & Q_2(1/2) &= 0. \end{aligned} \tag{12}$$

3. THE OPTIMALITY CRITERION

Let us assume the total volume of the column to be a cost function. We are looking for its minimum, e.g.

$$V = F_0 l \int_0^1 \Phi(x)\,\mathrm{d}x \rightarrow \min \tag{13}$$

subject to the constraints:

(a) for the prescribed external load associated with two different modes of buckling

$$\beta_1 = \beta_2 = \text{const.} \tag{14}$$

(b) for the critical time

$$t = t_* = \text{const.} \tag{15}$$

(c) for the cross-sectional area (additional geoemtrical constraint)

$$\Phi_1 \leqq \Phi(x) \leqq \Phi_2 \tag{16}$$

(d) fulfillment of the state eqns (10) with (11) and (12). The above problem will be solved with the use of the Pontryagin maximum principle formalism.

The Hamiltonian can be written as:

$$H = \sum_{i/1}^{2} k_i \left\{ \psi_{y_i}\varphi_i + \psi_{\varphi_i}\left[-\frac{M_i}{\Phi^{\nu}}\left(1 + T\left(\frac{\beta_i}{\Phi}\right)^{(n-\alpha-1)/(\alpha+1)}\right)\right] + \psi_{M_i}(Q_i + \beta_i\varphi_i) \right\} + \psi_0\Phi. \tag{17}$$

In our case the vector of Lagrangian multipliers $(\psi_{y_i}, \psi_{\varphi_i}, \psi_{M_i}, \psi_{Q_i})$ satisfies the same set of differential equations as the vector of state variables $(y_i, \varphi_i, M_i, Q_i)$.

Moreover, the boundary conditions of the adjoint set of equations are the same as (11) and (12). It means that the set of the state eqns (10) is selfadjoint.

Thus the Hamiltonian (17) can be now rewritten as:

$$H = \sum_{i/1}^{2} k_i \left\{ Q_i\varphi_i + \frac{M_i^2}{\Phi^{\nu}}\left[1 + T\left(\frac{\beta_i}{\Phi}\right)^{(n-\alpha-1)/(\alpha+1)}\right] + \varphi_i(Q_i + \beta_i\varphi_i) \right\} + \psi_0\Phi. \tag{18}$$

A necessary optimality condition $\partial H/\partial\Phi = 0$ leads to the following transcendental equation:

$$M_2^2\left[-\nu+\left(-\frac{n}{\alpha+1}-\nu+1\right)T\left(\frac{\beta_2}{\Phi}\right)^{(n-\alpha-1)/(\alpha+1)}\right]$$
$$+\mu M_1^2\left[-\nu+\left(-\frac{n}{\alpha+1}-\nu+1\right)T\left(\frac{\beta_1}{\Phi}\right)^{(n-\alpha-1)/(\alpha+1)}\right]+\lambda\Phi^{\nu+1}=0 \quad (19)$$

where $\mu = k_1/k_2$, $\lambda = \psi_0/k_2$.

4. DUAL FORMULATION

According to the method of solution applied (10) and (11), we reformulate the problem (13)–(16) as follows: we want to find such a cross-sectional area function $\Phi(x)$, under constraints:

$$t = t_* = \text{const.} \quad (20)$$

$$\int_0^1 \Phi(x)\,\mathrm{d}x = 1 \quad (21)$$

$$\Phi_1 \leq \Phi(x) \leq \Phi_2 \quad (22)$$

which simultaneously satisfies the state eqns (10) with (11) or (12) and maximizes a fundamental buckling load, i.e.

$$\beta \rightarrow \text{maximum}. \quad (23)$$

Such an approach is regarded as unimodal and for some values of Φ_1 it guarantees the higher values of the second (antisymmetric) buckling load. But, for some other values of Φ_1 the optimal shape $\Phi(x)$ has the lower value of the second buckling load. Thus, the obtained solution is actually not optimal. In this case, both symmetric and antisymmetric modes should be taken simultaneously into account. So, instead of (23) we have:

$$\beta_1 = \beta_2 \rightarrow \text{maximum} \quad (24)$$

with (20)–(22), (10), (11) and (12).

This approach is regarded as bimodal and the geometrical constraint can be then inactive (see paragraph 5 or [7, 12–14] for more detailed explanation).

The results have been obtained in an iterative way. The full numerical procedure of successive iteration is described in [10] for somewhat similar problems. The optimal distribution $\Phi(x)$, in the unimodal case, is obtained from (19) when $\mu = 0$. The constant λ should be calculated in accordance with (21). In the bimodal case the parameter μ is not equal to zero and its value is found numerically so that $\beta_1 = \beta_2$ is satisfied.

5. RESULTS

The numerical calculations were performed for $A = 2.18\ 10^{-113}$, $\alpha = 9.52$; $n = 32.8$/copper, temperature 200°C, [7]/.

Figure 1 gives the antisymmetric ($i = 1$) and symmetric ($i = 2$) modes of buckling for a prismatic column as a function of time t_* or the parameter T.

The maximal critical loads for a prescribed time t_* are given in the Table 1.

$t_*[h]$	0	1	24	720	8760	87600
β_{max}	52.51	7.76	7.07	6.42	5.95	5.57
$\beta_{prismatic}$	39.48	6.85	6.26	5.67	5.28	4.94

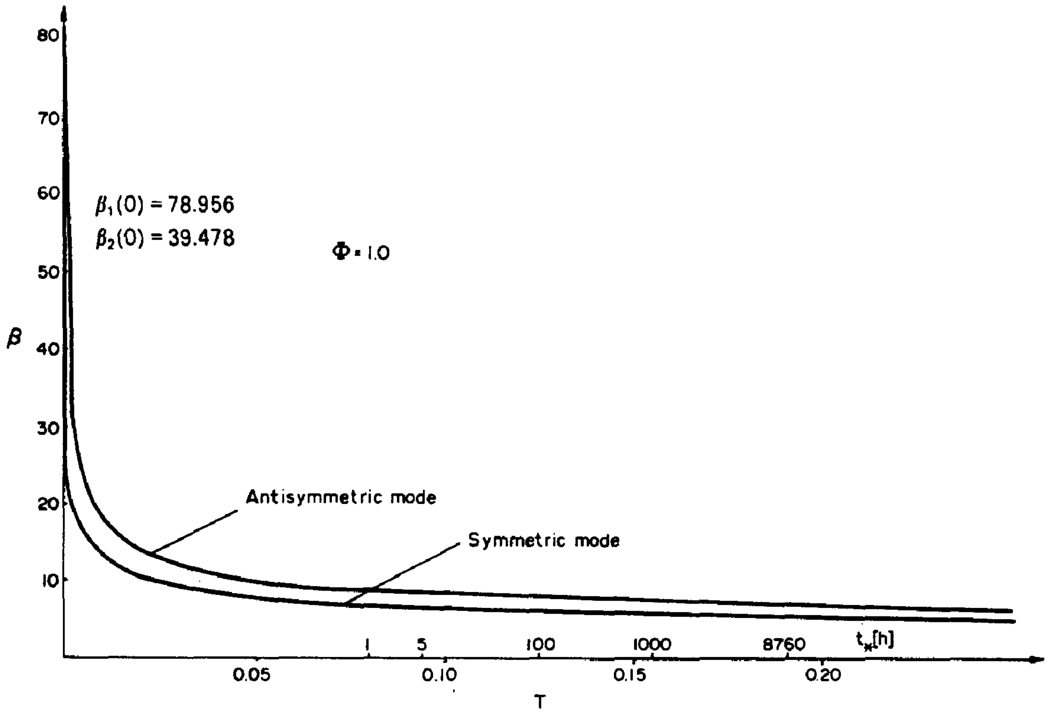

Fig. 1.

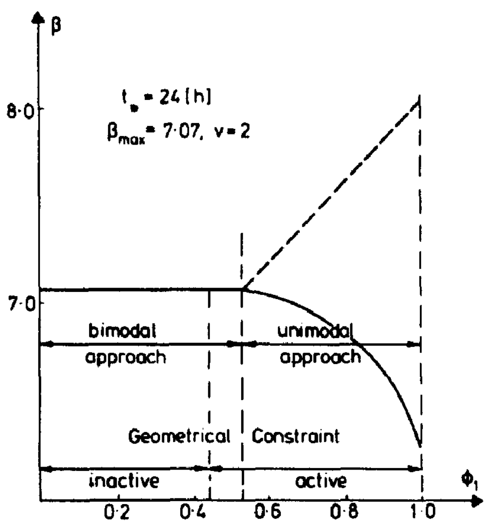

Fig. 2.

Figure 2 gives the maximal values of buckling load obtained in the maximization process of the first symmetric mode for the prescribed value of $t_* = 24$ [h] and $\Phi_1 \geq 0.53$. The buckling load of antisymmetric mode for the received optimal shapes is calculated and marked by dashed line. The corresponding optimal mass distribution for $\Phi_1 = 0.8$ is shown in Fig. 3. For $\Phi_1 < 0.53$ the bimodal approach has been required. As one can see (Fig. 3) the minimum constraint $\Phi_1 = 0.8$ is active for $0.203 \leq x \leq 0.303$ and $0.697 \leq x \leq 0.797$. Decreasing the minimum constraint we reach the point $\Phi_1 = 0.53$, (Fig. 2), where the minimum constraint is still active (N.B. for a shorter range of x). Any further decrease in Φ_1 gives erroneous solutions because the buckling load associated with the antisymmetric

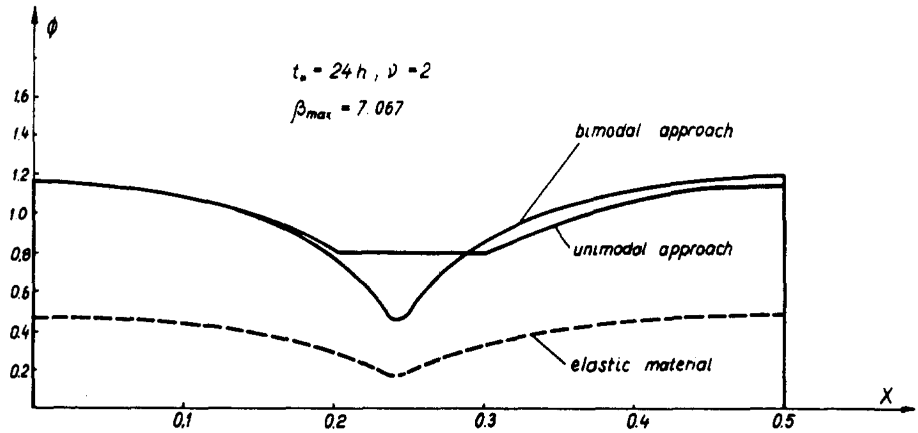

Fig. 3.

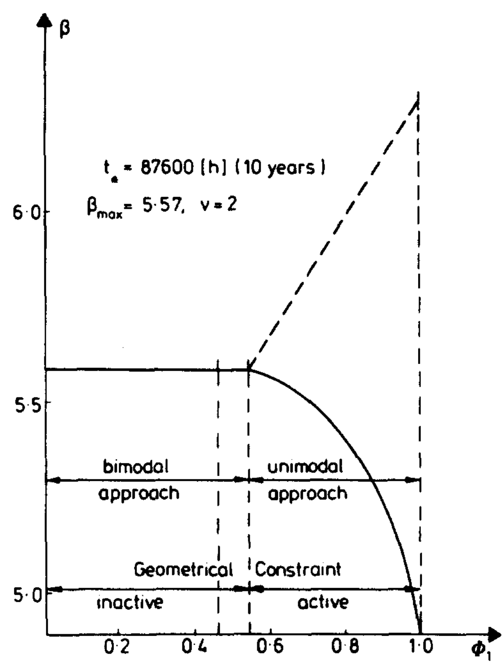

Fig. 4.

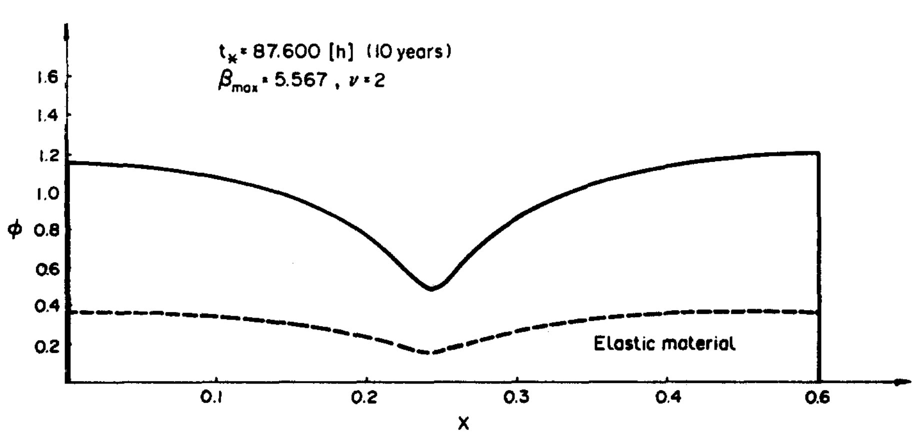

Fig. 5.

mode is lower than previously found in the unimodal optimization process. So, starting from $\Phi_1 = 0.53$ the optimal mass distribution is taken from eqn (19) with $\mu \neq 0$ (bimodal approach). The value of μ is chosen numerically in such a way as to obtain simultaneously the same values of the buckling load for $i = 1$ and $i = 2$. One observation should be made: As the minimum constraint Φ_1 decreases (in the range $\Phi_1 < 0.53$) the buckling load increases slightly and starting from a certain threshold value (in this case $\Phi_1 = 0.44$ in Fig. 2) the minimum constraint becomes inactive. In fact the optimal solution in this case does not depend on Φ_1 (see also [7, 12–14]). Appropriate optimal mass distribution is shown in Fig. 3, where optimal mass distribution of an elastic column, which has the same buckling load, is added. The same behaviour is observed in Figs. 4 and 5 for $t_* = 87600$ h.

6. CONCLUSIONS

Optimal shape of a clamped–clamped column under creep conditions is similar to that obtained in the elastic range, only the minimal cross-sections are subject to a more marked increase.

REFERENCES

1. F. R. Shanley, *Weight-Strength Analysis of Aircraft Structures*. McGraw–Hill, New York (1952).
2. G. Gerard, A creep buckling hypothesis. *J. Aero. Sci.* **23**, 879–882 (1956).
3. G. N. Rabotnov and S. A. Shesterikov, Creep stability of columns and plates. *J. Mech. Phys. Solids* **6**, 27–34 (1957).
4. W. E. Jahsman and F. A. Field, Creep instability and collapse of initially straight columns. Lockheed Aircraft Co., Rep. No. 2458, Sunnyvale, California (June 1958).
5. N. J. Hoff, A survey of the theories of creep buckling. *Proc. 3rd U.S. Nat. Congr. Appl. Mech.* pp. 29–49. Pergamon Press, Oxford (1958).
6. N. J. Hoff, Reversed creep: a remark to the creep buckling theory of Rabotnov and Shesterikov. *J. Mech. Phys. Solids* **12**, 113–123 (1964).
7. M. Życzkowski and R. Wojdanowska-Zając, Optimal structural design with respect to creep buckling. *Proc. IUTAM Symp. Creep in Structures II*, pp. 371–387. Springer–Verlag, Berlin (1972).
8. J. Kiusalaas, Optimal design of structures with buckling constraints. *Int. J. Solids Structures* **9**, 863–878 (1973).
9. N. Olhoff and S. H. Rasmussen, On single and bimodal optimum buckling loads of clamped columns. *Int. J. Solids Structures* **13**, 605–614 (1977).
10. J. Błachut and A. Gajewski, A unified approach to optimal design of columns. *Solid. Mech. Arch.* **5**, 363–413 (1980).
11. J. Błachut, A note on optimal design of a beam with a mass at its end. *J. Sound Vib.* **80**, 203–208 (1982).
12. J. Błachut and A. Gajewski, On unimodal and bimodal optimal design of funicular arches. *Int. J. Solids Structures* **17**, 653–667 (1981).
13. J. Błachut and A. Gajewski, Unimodal and bimodal optimal design of extensible arches with respect to buckling and vibrations. *Optimal Contr. Appl. Meth.* **2**, 383–402 (1981).
14. B. Bochenek and A. Gajewski, Certain problems of unimodal and bimodal optimal design of structures, optimization meth. in struct. design. *Proc.* 164 *Euromech Coll.* (Edited by H. Eschenauer and N. Olhoff), pp. 204–209 (1982).